

Newton's Constant isn't constant¹

Martin Reuter

University of Mainz, Germany

According to Newton's law of gravity, two masses interact with each other via a central force which can be derived from the potential $V_N(r) = -Gm_1m_2/r$ where G is a universal constant known as Newton's constant. Likewise, according to Coulomb's law, two electrically charged particles with charges n_1e and n_2e interact via the central potential $V_{Cb}(r) = \alpha n_1n_2/r$ where e is the elementary charge and where $\alpha \equiv e^2/4\pi$ is nowadays referred to as the fine structure constant. Thus, in classical physics, the gravitational and the electrostatic interactions are described by exactly the same $1/r$ -law, and their respective coupling strengths are determined by the two universal constants G and α which enter the equations in an analogous fashion.

However, from the point of view of modern quantum field theory we know that e and α are not really constants but are more appropriately considered scale dependent or "running" quantities. In quantum electrodynamics (QED) the charge e of a positron is a function $e(k)$ depending on the "renormalization scale" k , a parameter with the dimension of a mass which specifies the resolution of the "microscope" with which we probe the system. The physical mechanism behind the scale dependence of the electric charge is easy to understand. The combination of Quantum Mechanics and Special Relativity converts the vacuum of electrodynamics into a sea of virtual electron-positron pairs which are continuously created and annihilated. When we immerse an external test charge into this sea it gets polarized in very much the same way as an ordinary dielectric. The polarization cloud of the virtual e^+/e^- -pairs surrounding the test charge tends to screen it, and it appears to be larger at small distances and smaller at large distances. In an experiment which resolves length scales $\ell \equiv k^{-1}$ one measures the effective charge $e(k)$ which includes the effect of this polarization of the vacuum.

As a consequence of the same screening mechanism the classical Coulomb potential is replaced by a more complicated quantum corrected potential, the Uehling potential $V_{Uehling}(r)$. At least in the limit of massless electrons, this potential is directly related to the running charge. Considering an electron in the field of a positron, say, one starts from the classical potential energy $V_{Cb}(r) = -e^2/4\pi r$ and replaces e^2 by the running gauge coupling in the one-loop approximation:

$$e^2(k) = e^2(k_0) [1 - b \ln(k/k_0)]^{-1}, \quad b \equiv e^2(k_0)/6\pi^2.$$

(We are using units such that $\hbar = c = 1$.) The crucial step is to identify the renormalization point k with the inverse of the distance r . This is possible because

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in the massless theory r is the only dimensionful quantity which could define a scale. The result of this substitution reads

$$V_{\text{Uehling}}(r) = -e^2(r_0^{-1}) [1 + b \ln(r_0/r) + O(e^4)] / 4\pi r$$

where the IR reference scale $r_0 \equiv 1/k_0$ has to be kept finite in the massless theory. Our result is the correct (one-loop, massless) Uehling potential which is usually derived from the polarization tensor of the photon. Obviously the position dependent renormalization group improvement $e^2 \rightarrow e^2(k)$, $k \propto 1/r$ encapsulates the most important effects the quantum fluctuations have on the electric field produced by a point charge.

Because of the analogy between α and G it is natural to ask if there are similar quantum effects which render Newton's constant scale dependent. Clearly the first step towards an answer to this question consists of replacing Newtonian gravity by General Relativity. Here the relevant field-source relation is Einstein's equation

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu}$$

which reproduces Newton's law in an appropriate limit. In General Relativity, too, G is a universal constant, the coupling constant of the gravitational self-interaction and of the gravity-matter interaction.

Contrary to the situation in electrodynamics we have no consistent fundamental quantum field theory of gravity at our disposal yet. Nevertheless, guided by the analogy with the running electric charge, it is tempting to speculate on how quantum gravitational effects might modify Newton's law and lead to a scale dependence of G . It is plausible to assume that in the large distance limit the leading quantum effects are described by quantizing the linear fluctuations of the metric, $g_{\mu\nu}$. One obtains a free field theory in a possibly curved background spacetime whose elementary quanta, the gravitons, carry energy and momentum. The vacuum of this theory will be populated by virtual graviton pairs, and the central question is how these virtual gravitons respond to the perturbation by an external test body which we immerse in the vacuum. Assuming that also in this situation gravity is universally attractive, the gravitons will be attracted towards the test body. Hence it will become "dressed" by a cloud of virtual gravitons surrounding it so that its effective mass seen by a distant observer is larger than it would be in absence of any quantum effects. This means that while in QED the quantum fluctuations *screen* external charges, in quantum gravity they have an *antiscreening* effect on external test masses. This entails Newton's constant becoming a scale dependent quantity $G(k)$ which is small at small distances $\ell \equiv k^{-1}$, and which becomes large at larger distances. This behavior is similar to the running of the nonabelian gauge coupling in Yang-Mills Theory.

Can we verify these heuristic arguments within a consistent theory? In many of the traditional approaches to quantum gravity the Einstein-Hilbert term $\int d^4x \sqrt{-g} R$ has been regarded as a fundamental action which should be quantized along the same lines as the familiar renormalizable field theories in flat space, such as QED for example. It was soon realized that this program is not only technically rather involved but also leads to severe conceptual difficulties. In particular, the nonrenormalizability of the theory hampers a meaningful perturbative analysis. While this does not rule out the possibility that the theory exists nonperturbatively, not much is known in this direction. However, it could also be argued that gravity, as we know it, should not be quantized at all, because Einstein gravity is an effective theory which results from quantizing some yet unknown fundamental theory. If so, the Einstein-Hilbert term is an effective action analogous to the Heisenberg-Euler action in QED, say, and it should not be compared to the “microscopic” action of electrodynamics.

It seems not unreasonable to assume that the truth lies somewhere between those two extreme points of view, i.e., that Einstein gravity is an effective theory which is valid near a certain nonzero momentum scale k . This means that it arises from the fundamental theory by a “partial quantization” in which only excitations with momenta larger than k are integrated out, while those with momenta smaller than k are not included. (The interpretation of the Einstein-Hilbert term as a fundamental or an ordinary effective action is recovered in the limits $k \rightarrow \infty$ and $k \rightarrow 0$, respectively.) By definition, an “effective theory at scale k ”, when evaluated at tree level, should correctly describe all gravitational phenomena which involve a typical momentum scale k acting as a physical infrared cutoff. Only if one is interested in processes with momenta $k' \ll k$, loop calculations become necessary; they amount to integrating out the missing field modes in the momentum interval $[k', k]$.

In ref.[1] it was proposed to regard the scale-dependent action for gravity, henceforth denoted $\Gamma_k[g_{\mu\nu}]$ (“effective average action”), as a Wilsonian effective action which is obtained from the fundamental (“microscopic”) action S by a kind of coarse-graining analogous to the iterated block-spin transformations which are familiar from lattice systems. In the continuum, Γ_k is defined in terms of a modified Euclidean functional integral over e^{-S} in which the contributions of all field modes with momenta smaller than k are suppressed. In this manner Γ_k interpolates between S (for $k \rightarrow \infty$) and the standard effective action Γ (for $k \rightarrow 0$). The trajectory in the space of all action functionals can be obtained as the solution of a certain functional evolution equation, the exact renormalization group (RG) equation. Its form is independent of the action S under consideration. The latter enters via the initial conditions for the renormalization group trajectory; it is specified at some UV cutoff scale Λ : $\Gamma_\Lambda = S$. If S is a *fundamental* action, Λ is sent to infinity at the end. The renormalization group equation

can also be used to evolve *effective* actions, known at some point Λ , towards smaller scales $k < \Lambda$. In this case Λ is a fixed, finite scale. In this framework, the (non)renormalizability of a theory is seen as a global property of the renormalization group flow for $\Lambda \rightarrow \infty$. The evolution equation by itself is perfectly finite and well behaved in either case because it describes only infinitesimal changes of the cutoff.

In the construction of ref.[1] the modified functional integral over e^{-S} is similar to the standard gauge-fixed path-integral of Euclidean gravity in the background gauge. The crucial new ingredient is a built-in infrared (IR) cutoff which suppresses the contributions from long-wavelength field modes. It is implemented by giving a k -dependent and mode-dependent mass $\mathcal{R}_k(p^2)$ to the modes with covariant momentum p . Inside loops, it suppresses the small- p contributions. The function $\mathcal{R}_k(p^2)$ has to satisfy $\mathcal{R}_k(p^2) \rightarrow 0$ for $k \rightarrow 0$ and $\mathcal{R}_k(p^2) \propto k^2$ for $k \gg p$, but is arbitrary otherwise. (In practice the exponential cutoff $\mathcal{R}_k(p^2) \propto p^2[\exp(p^2/k^2) - 1]^{-1}$ is convenient.)

In order to obtain a functional $\Gamma_k[g]$ which is invariant under general coordinate transformations the background gauge formalism is employed. This means that we actually RG-evolve an action $\Gamma_k[g, \bar{g}]$ which depends on both the “ordinary” metric $g_{\mu\nu}$ and on a background metric $\bar{g}_{\mu\nu}$. The standard action is recovered by setting $\bar{g} = g$, i.e. $\Gamma_k[g] \equiv \Gamma_k[g, g]$. The exact renormalization group equation for $\Gamma_k[g, \bar{g}]$ reads (see ref.[1] for details):

$$\begin{aligned} \partial_t \Gamma_k[g, \bar{g}] = & \frac{1}{2} \text{Tr} \left[\left(\kappa^{-2} \Gamma_k^{(2)}[g, \bar{g}] + \mathcal{R}_k^{\text{grav}}[\bar{g}] \right)^{-1} \partial_t \mathcal{R}_k^{\text{grav}}[\bar{g}] \right] \\ & - \text{Tr} \left[\left(-\mathcal{M}[g, \bar{g}] + \mathcal{R}_k^{\text{gh}}[\bar{g}] \right)^{-1} \partial_t \mathcal{R}_k^{\text{gh}}[\bar{g}] \right] \end{aligned}$$

with the “renormalization group time” $t \equiv \ln k$. Here $\Gamma_k^{(2)}$ stands for the Hessian of Γ_k with respect to $g_{\mu\nu}$ at fixed $\bar{g}_{\mu\nu}$, and \mathcal{M} is the Faddeev-Popov ghost operator. The operators $\mathcal{R}_k^{\text{grav}}$ and $\mathcal{R}_k^{\text{gh}}$ implement the IR cutoff in the graviton and the ghost sector. They obtain from $\mathcal{R}_k(p^2)$ by replacing the momentum square p^2 with the graviton and ghost kinetic operator, respectively.

Nonperturbative solutions to the above RG-equation (which do not require any expansion in G) can be obtained by the method of “truncations”. This means that one projects the RG flow $k \mapsto \Gamma_k$ in the infinite dimensional space $\{\Gamma[\cdot]\}$ of all action functionals onto some finite dimensional subspace which is particularly relevant. In this manner the functional RG-equation becomes an ordinary differential equation for a finite set of generalized couplings which serve as coordinates on this subspace. In ref.[1] we projected on the 2-dimensional subspace spanned by the operators \sqrt{g} and $\sqrt{g}R$ (“Einstein-Hilbert truncation”). This truncation

of the “theory space” amounts to considering only actions of the form

$$\Gamma_k[g, \bar{g}] = (16\pi G(k))^{-1} \int d^4x \sqrt{\bar{g}} \{-R(g) + 2\bar{\lambda}(k)\} + S_{\text{gf}}[g, \bar{g}]$$

where $G(k)$ and $\bar{\lambda}(k)$ denote the running Newton constant and cosmological constant, respectively, and where S_{gf} is the classical background gauge fixing term. More general (and, therefore, more precise) truncations would include higher powers of the curvature tensor as well as nonlocal terms, for instance.

In ref.[1] we inserted the Einstein-Hilbert ansatz into the RG-equation and derived the coupled system of equations for $G(k)$ and $\bar{\lambda}(k)$. It is rather complicated and we shall not write it down here. If $\bar{\lambda} \ll k^2$ for all scales of interest, it simplifies considerably and boils down to a simple equation for the dimensionless Newton constant $g(k) \equiv k^2 G(k)$ [2]:

$$\frac{d}{dt}g(t) = \beta(g(t)), \quad \beta(g) = 2g \frac{1 - \omega'g}{1 - B_2g}$$

For the exponential cutoff, the constants entering the beta-function are [2]:

$$\omega' \equiv \omega + B_2, \quad \omega = \frac{4}{\pi} \left(1 - \frac{\pi^2}{144}\right), \quad B_2 = \frac{2}{3\pi}$$

The above evolution equation for g displays two fixed points $g_*, \beta(g_*) = 0$. There exists an infrared attractive (gaussian) fixed point at $g_*^{\text{IR}} = 0$ and an ultraviolet attractive (nongaussian) fixed point at

$$g_*^{\text{UV}} = \frac{1}{\omega'}$$

This latter fixed point is a higher dimensional analog of the Weinberg fixed point [3] known from $(2 + \epsilon)$ -dimensional gravity.

The UV fixed point separates a weak coupling regime ($g < g_*^{\text{UV}}$) from a strong coupling regime where $g > g_*^{\text{UV}}$. Since the β -function is positive for $g \in [0, g_*^{\text{UV}}]$ and negative otherwise, the renormalization group trajectories $k \mapsto g(k)$ fall into the following three classes:

- (i) Trajectories with $g(k) < 0$ for all k . They are attracted towards g_*^{IR} for $k \rightarrow 0$.
- (ii) Trajectories with $g(k) > g_*^{\text{UV}}$ for all k . They are attracted towards g_*^{UV} for $k \rightarrow \infty$.
- (iii) Trajectories with $g(k) \in [0, g_*^{\text{UV}}]$ for all k . They are attracted towards $g_*^{\text{IR}} = 0$ for $k \rightarrow 0$ and towards g_*^{UV} for $k \rightarrow \infty$.

Only the trajectories of type (iii) are relevant for us. We shall not allow for a negative Newton constant, and we also discard solutions of type (ii). They are in the strong coupling region and do not connect to a perturbative large distance regime. (See ref.[4] for a detailed numerical investigation of the phase diagram.)

The trajectories of type (iii) cannot be written down in closed form but, returning to the dimensionful quantity G , a numerically rather precise approximation is given by

$$G(k) = \frac{G(k_0)}{1 + \omega G(k_0) [k^2 - k_0^2]}$$

We shall set $k_0 = 0$ for the reference scale. At least within the Einstein-Hilbert truncation, $G(k)$ does not run any more between scales where the Newton constant was determined experimentally (laboratory scale, scale of the solar system, etc.) and $k \approx 0$ (cosmological scale). Therefore we can identify $G_0 \equiv G(k_0 = 0)$ with the experimentally observed value of the Newton constant. From

$$G(k) = \frac{G_0}{1 + \omega G_0 k^2}$$

we see that when we go to higher momentum scales k , $G(k)$ decreases monotonically. For small k we have

$$G(k) = G_0 - \omega G_0^2 k^2 + O(k^4)$$

while for $k^2 \gg G_0^{-1}$ the fixed point behavior sets in and $G(k)$ “forgets” its infrared value:

$$G(k) \approx \frac{1}{\omega k^2}$$

In ref.[5], Polyakov had conjectured an asymptotic running of precisely this form. If the UV fixed point can be confirmed by more general truncations it means that Einstein gravity in 4 dimensions is “asymptotically safe” in Weinberg’s sense [3].

Thus the heuristic arguments in favor of the antiscreening character of pure quantum gravity seem to be correct [8], and we may now use our result for $G(k)$ in order to RG-improve Newton’s potential. The leading large distance correction of $V_N(r)$ is obtained by using the small- k approximation for $G(k)$ and by setting $k = \xi/r$, $\xi = \text{const}$, because $1/r$ is the only relevant IR cutoff if spacetime is approximately flat. Reinstating factors of \hbar and c for a moment we find

$$V_{\text{imp}}(r) = -G_0 \frac{m_1 m_2}{r} \left[1 - \tilde{\omega} \frac{G_0 \hbar}{r^2 c^3} + \dots \right]$$

The constant $\tilde{\omega} \equiv \omega \xi^2$ is predicted to be positive, but its precise value cannot be inferred from RG-arguments alone. However, it was pointed out by Donoghue [6]

that the standard perturbative quantization of Einstein gravity leads to a well-defined, finite prediction for the leading large distance correction to Newton's potential. His result reads

$$V(r) = -G_0 \frac{m_1 m_2}{r} \left[1 - \frac{G_0 (m_1 + m_2)}{2c^2 r} - \hat{\omega} \frac{G_0 \hbar}{r^2 c^3} + \dots \right]$$

where $\hat{\omega} = 118/15\pi$. The correction proportional to $(m_1 + m_2)/r$ is a purely kinematic effect of classical general relativity, while the quantum correction $\propto \hbar$ has precisely the structure we have predicted on the basis of the renormalization group. Comparing the two potentials allows us to determine the coefficient $\tilde{\omega}$ by identifying $\tilde{\omega} = \hat{\omega}$.

With its only undetermined parameter fixed by Donoghue's asymptotic calculation, we can now use our formula for $G(r) \equiv G(k(r))$ in order to investigate gravity at very short distances comparable to the Planck length. In refs. [2, 7] the impact of the running of G on the structure of black holes has been considered as an example. In ref.[2] we constructed a quantum-Schwarzschild black hole by improving $G_0 \rightarrow G(r)$ in the classical Schwarzschild metric. (In this context the correct identification of k as a function of r is more subtle; a careful analysis yields $k \propto r^{-1}$ for $r \rightarrow \infty$, but $k \propto r^{-3/2}$ for $r \rightarrow 0$.) The main features of the RG improved spacetime are as follows.

As far as the structure of horizons is concerned, the quantum effects are small for very heavy black holes ($M \gg m_{\text{Pl}}$). They have an event horizon at a radius r_+ which is close to, but always smaller than the Schwarzschild radius $2G_0 M$. Decreasing the mass of the black hole the event horizon shrinks. There is also an inner (Cauchy) horizon whose radius r_- increases as M decreases. When M equals a certain critical mass M_{cr} which is of the order of the Planck mass the two horizons coincide. The near-horizon geometry of this extremal black hole is that of $AdS_2 \times S^2$. For $M < M_{\text{cr}}$ the spacetime has no horizon at all.

While the exact fate of the singularity at $r = 0$ cannot be decided within our present approach, it can be argued that either it is not present at all or it is at least much weaker than its classical counterpart. In the first case the quantum spacetime has a smooth de Sitter core so that we are in accord with the cosmic censorship hypothesis even if $M < M_{\text{cr}}$.

The conformal structure of the quantum black hole is very similar to that of the classical Reissner-Nordström spacetime. In particular its $(r = 0)$ -hypersurface is timelike, in contradistinction to the Schwarzschild case where it is spacelike.

The Hawking temperature T_{BH} of very heavy quantum black holes is given by the semiclassical $1/M$ -law. As M decreases, T_{BH} reaches a maximum at $\tilde{M}_{\text{cr}} \approx 1.27 M_{\text{cr}}$ and then drops to $T_{\text{BH}} = 0$ at $M = \tilde{M}_{\text{cr}}$. The specific heat capacity has a singularity at \tilde{M}_{cr} . It is negative for $M > \tilde{M}_{\text{cr}}$, but positive for $\tilde{M}_{\text{cr}} > M > M_{\text{cr}}$.

We argued that the vanishing temperature of the extremal black hole leads to a termination of the evaporation process once the black hole has reduced its mass to $M = M_{\text{cr}}$. This supports the idea of a cold, Planck size remnant as the final state of the black hole evaporation.

For $M > M_{\text{cr}}$, the entropy of the quantum black hole is a well defined, monotonically increasing function of the mass. For heavy black holes we recover the classical expression $\mathcal{A}/4G_0$. The leading quantum corrections are proportional to $\ln(M/M_{\text{cr}})$.

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